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Hamilton's principle for systems of changing mass

D. B. McIVER

Department of Shipbuilding and Naval Architecture, University of Strathclyde, Glasgow, Scotland, U.K.

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SUMMARY

An extended form of Hamilton's principle is developed for systems whose constituent particles change with time. By a suitable choice of system boundary it is demonstrated that in some cases stationary forms of the principle are possible. A simple example is considered.

1. Introduction

Hamilton's principle, as usually formulated, is valid only for systems composed always of the same particles. There exist only a few isolated attempts to extend it to the analysis of systems whose constituent particles change with time. Furthermore, these are restricted to the problem of the flow of fluid through flexible or articulated pipes which, although there is material transported through the system, maintain a constant mass.

This paper demonstrates that, by a quite simple modification, the Hamiltonian principle for a material, or closed, system of fixed identity and constant mass can be extended to open systems of changing mass. The open system boundary is quite arbitrary and consequently it is possible to choose its motion in the manner best suited to the problem in hand. In particular, it becomes feasible to define open systems having constant total energy despite the presence of non-conservative forces, or, provided certain rather restrictive conditions are satisfied, an open system which is characterised by a stationary form of Hamilton's principle. Unhappily with the open system, unlike the conservative closed system, the conservation of energy is not a consequence of the absence of non-conservative forces.

A discussion of existing work is delayed until after the development of the general theory since it yields an independent verification and clarification of the former. The method is applicable to a wide range of problems but herein the illustration of the method is restricted to the rectilinear flight of a rocket in a vacuum.

2. The Hamiltonian principle for a closed system

Consider a continuous material system of particles of fixed identity contained within a moving region of space $R_c(t)$ bounded by the surface $B_c(t)$ across which there is no mass transport at any point. At position $\boldsymbol{r}$ at time t the particle density is ρ and the velocity $\boldsymbol{u}$.

It is well known [1,2,4] that the principle of virtual work for the system may be written quite generally as

$$\delta(L)_c + \delta W - \frac{D}{Dt}\iiint_{R_c(t)} \rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})\,dv = 0 \tag{1}$$

when the system is subjected to a virtual displacement distribution $\delta\boldsymbol{r}=\delta\boldsymbol{r}(\boldsymbol{r}, t)$ satisfying any imposed constraints. These are discussed more fully in Section 5 but for the present include the satisfaction of the balance laws of mass, internal energy and entropy within the system together with any geometrical and thermodynamic constraints at the boundary.

$(L)_c=(K-E)_c$ is the Lagrangian in which $(K)_c$ is the kinetic energy of the particles within the closed region $R_c(t)$ at any instant and $(E)_c$ a corresponding potential energy. δW is the virtual work performed by the generalized forces undergoing generalized virtual displacements.

$D(\ )/Dt$ is the material time derivative following a particle or a specific collection of particles; thus, $\boldsymbol{u} = D\boldsymbol{r}/Dt$.

If, for example, the system is a non-conducting linear elastic solid the potential energy $(E)_c$ may be identified as the sum of the strain energy and the gravitational potential energy. The virtual work is then the sum of contributions from the non-potential body forces and surface tractions undergoing the virtual displacements $\delta\boldsymbol{r}$. Where thermodynamic effects are included the strain energy becomes the internal energy and the virtual work acquires contributions due to virtual temperature changes at the boundary [3].

Hamilton's principle is obtained by integrating (1) with respect to time between two instants t_1 and t_2, when the system configuration is prescribed ($\delta\boldsymbol{r} = 0$). Thus, since variations to the time are not permitted,

$$\delta \int_{t_1}^{t_2} (L)_c dt + \int_{t_1}^{t_2} \delta W dt = 0 . \tag{2}$$

Any system with a finite number of discontinuities may be treated as a set of continuous subsystems to each of which the principle (2) can be applied. When recombined and the balance law jump conditions imposed across the discontinuities it is quite simple to show that the above form of Hamilton's principle is still applicable. Thus, we can use the principle to investigate, for instance, the dynamics of combined fluid-solid systems. Later, in Section 8, the equation of motion for the rectilinear flight of a rocket is derived but to treat this variable mass system requires modification of the principle.

3. The open system

To extend the principle to a system of changing mass the open control volume $R_o(t)$ is introduced with a portion $B_c(t)$ of its complete boundary $B_c(t) + B_o(t)$ capable of movement with an, as yet, arbitrary normal velocity $\boldsymbol{V} \cdot \boldsymbol{n}$ and across which mass is transported. $\boldsymbol{n}$ is the outward normal.

At all instants t the system is defined as the collection of particles inhabiting the open control volume $R_o(t)$. The control volume is pervious to particles and the system is thus not necessarily of constant mass or, if of constant mass, it need not consist always of the same set of particles. The region $R_c(t)$ mentioned earlier is thus a closed control volume bounded by the closed control surface $B_c(t)$ for which $\boldsymbol{V} \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \boldsymbol{n}$.

If, at the instant considered, the open control volume $R_o(t)$ coincides with the closed control volume $R_c(t)$, the general transport theorem reads as

$$\frac{d}{dt} \iiint_{R_o(t)} (\ ) dv = \frac{D}{Dt} \iiint_{R_c(t)} (\ ) dv + \iint_{B_o(t)} (\ )(\boldsymbol{V} - \boldsymbol{u}) \cdot \boldsymbol{n}\, ds . \tag{3}$$

The motions of the separate regions are illustrated below in Figure 1.

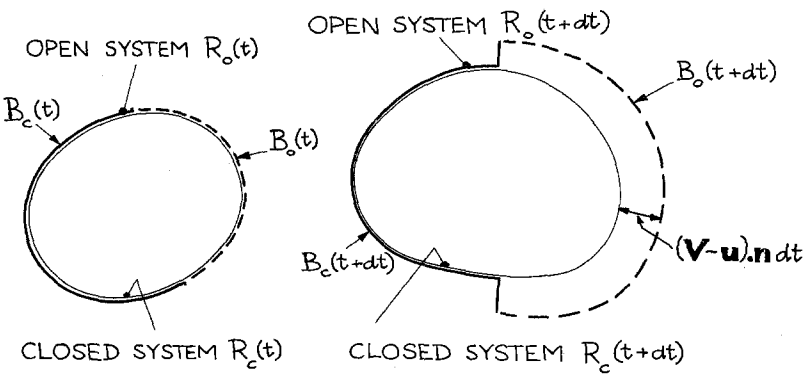

Figure 1.

It is permissible to write

$$\frac{D}{Dt}\iiint_{R_c(t)}(\,)dv = \frac{D}{Dt}\iiint_{R_o(t)}(\,)dv \tag{4}$$

since the use of the special notation $D(\,)/Dt$ makes it clear that a closed control volume is to be employed.

4. The extended Hamiltonian principle for a system of changing mass

Employing the general transport theorem (3), the virtual work equation (1) may be rewritten as

$$\delta(L)_0+\delta W+\iint_{B_o(t)}\rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds-\frac{d}{dt}\iiint_{R_o(t)}\rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})dv=0 \tag{5}$$

and then, integrating with respect to time as before between t_1 and t_2 when the system configuration is prescribed, the extended form of Hamilton's principle for a system of changing mass becomes

$$\delta\int_{t_1}^{t_2}(L)_0\,dt+\int_{t_1}^{t_2}\delta H\,dt=0 \tag{6}$$

where,

$$\delta H=\delta W+\iint_{B_o(t)}\rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds\,. \tag{7}$$

$(L)_0=(K-E)_0$ is the Lagrangian of the system contained within the open control volume $R_o(t)$ and δW is the virtual work performed on this same system. The last integral on the right of (7) may be thought of as a virtual momentum transport across the open surface $B_o(t)$.

Having derived the extended principle in its most general form, it is now necessary to prepare the way for future discussion by being more specific in the details of the system.

In the remainder of the paper (except Section 6) only virtual work contributions are considered which arise from surface traction over the open and closed portions of the control volume boundary. The virtual work may then be conveniently written as

$$\begin{aligned}\delta W&=\delta W_c+\delta W_0\\&=\iint_{B_c(t)}\delta\boldsymbol{r}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\,ds+\iint_{B_o(t)}\delta\boldsymbol{r}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\,ds\end{aligned} \tag{8}$$

where $\bar{\boldsymbol{\sigma}}$ is a symmetrical stress dyadic and $\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}$ is the surface traction vector. The surfaces $B_c(t)$ and $B_o(t)$ are recognised as perhaps having portions over which the particle positions are prescribed and which therefore make no contribution to the virtual work. The restriction imposed could well be on the type of virtual change admissible in the most general system as much as on the nature of the system itself. For instance, still included are systems in which thermodynamic changes are active but in which the thermodynamic characteristics (such as temperature) are prescribed on the boundary: but excluded are systems on which act non-potential body forces.

The extended Hamiltonian principle is now

$$\delta\int_{t_1}^{t_2}(K-E)_0\,dt+\int_{t_1}^{t_2}dt\iint_{B_c(t)}\delta\boldsymbol{r}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\,ds+\int_{t_1}^{t_2}dt\iint_{B_o(t)}\delta\boldsymbol{r}\cdot[\bar{\boldsymbol{\sigma}}+\rho\boldsymbol{u}(\boldsymbol{V}-\boldsymbol{u})]\cdot\boldsymbol{n}\,ds=0 \tag{9}$$

with

$$(K)_0=\iiint_{R_o(t)}\tfrac{1}{2}\rho u^2\,dv \tag{10}$$

and

$$(E)_0=\iiint_{R_o(t)}\rho e\,dv \tag{11}$$

where e is the potential energy per unit mass.

For a prescribed surface traction over $B_c(t)$ and $B_o(t)$ which is independent of position, the virtual work can be expressed as a total differential; thus,

$$\delta W = \delta \iint_{B_c(t)+B_o(t)} \boldsymbol{r}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\,ds = \delta(W_c+W_o) \tag{12}$$

and the variational principle becomes

$$\delta\int_{t_1}^{t_2}[(K-E)_0+W_c+W_o]\,dt + \int_{t_1}^{t_2} dt \iint_{B_o(t)} \rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})(V-\boldsymbol{u})\cdot\boldsymbol{n}\,ds = 0\,. \tag{13}$$

5. The variational process and the role of the constraints

Not all infinitesimal displacement distributions are permissible. Besides having to satisfy any imposed geometrical constraints, thereby ensuring that there is no virtual work contribution from the reactions, the virtual displacements must preserve the balance laws of mass, internal energy and entropy and obey any constitutive laws.

The mass balance law appears either in the local form of the continuity equation (for a continuum)

$$\frac{D}{Dt}(\rho\,dv) = 0 \tag{14}$$

or in the global, or integral, form for the complete system

$$\frac{d}{dt}\iiint_{R_o(t)} \rho\,dv = \iint_{B_o(t)} \rho(V-\boldsymbol{u})\cdot\boldsymbol{n}\,ds\,,\quad \frac{D}{Dt}\iiint_{R_o(t)} \rho\,dv = 0\,. \tag{15}$$

It further follows from (14) that

$$\frac{D}{Dt}\iiint_{R_o(t),\,R_c(t)} \rho(\,)\,dv = \iiint_{R_o(t),\,R_c(t)} \rho\frac{D}{Dt}(\,)\,dv\,. \tag{16}$$

The constraints may be satisfied by resorting to the method of Lagrange multipliers but the alternative used herein (at least for the mass constraint) is to employ the variational form of the constraint equation—but indirectly.

First, some consideration is necessary of the meaning of a variation to an integral over a region $R_o(t)$. The variation denoted by the operator $\delta(\,)$ is the first order change to the integral as the aggregate of particles contained within the region of integration, or control volume, suffers a virtual displacement. It is a material variation in the same way that the operator $D(\,)/Dt$ constitutes a material rate of change and so the two operators are interchangeable and also commute; that is, $\delta D(\,)/Dt = D\delta(\,)/Dt$. The control volume, open or closed, is always a closed system as far as the variation is concerned regardless of whether or not material is transported across its boundaries in the real motion: there is no virtual material transport out of the system.

Corresponding to (14), the variational form of local mass balance is

$$\delta(\rho\,dv) = 0\,. \tag{17}$$

It follows from this, or just by changing the operators in (16), that

$$\delta\iiint_{R_o(t),\,R_c(t)} \rho(\,)\,dv = \iiint_{R_o(t),\,R_c(t)} \rho\,\delta(\,)\,dv\,. \tag{18}$$

6. The energy equation and momentum balance laws consequent on the extended principle

By permitting the virtual displacements to coincide with actual displacements, the energy equation can be recovered from the virtual work equation. In general, if the system is in motion

at t_1 and t_2, the virtual displacements cannot now be zero at these instants. Thus, $\delta \boldsymbol{r} = \boldsymbol{u}\,dt$ and the variational operator is related to the time differential operator by $\delta(\ ) = dt\,D(\ )/Dt$.

As a consequence, equation (5) becomes, after cancellation of a factor dt,

$$\frac{D}{Dt}(K-E)_0 + \dot{W} + \iint_{B_o(t)} \rho u^2 (\boldsymbol{V}-\boldsymbol{u})\cdot \boldsymbol{n}\,ds - \frac{d}{dt}(2K)_0 = 0 \tag{19}$$

and then, by (10), (11) and the transport theorem (3), the energy equation is

$$\frac{d}{dt}(K+E)_0 = \dot{W} + \iint_{B_o(t)} \rho\left(\tfrac{1}{2}u^2 + e\right)(\boldsymbol{V}-\boldsymbol{u})\cdot \boldsymbol{n}\,ds\,. \tag{20}$$

$\dot{W}$ represents the rate at which work is being performed on the system by its surroundings; in general, it is not a total derivative. The second term on the right is the rate of gain of energy by virtue of the advancing control surface engulfing particles.

Hamilton's principle yields the balance laws of linear and angular momenta for the system when the virtual displacement takes the form of a rigid body virtual translation and rotation respectively. When $(E)_0$ represents the internal energy of the system, its variation under a rigid body virtual displacement will be zero: when it is the potential of a body force of some kind, its variation can be expressed as $\boldsymbol{P}\cdot\delta\boldsymbol{r}_G$ where $\boldsymbol{P}$ is the body force and $\delta\boldsymbol{r} = \delta\boldsymbol{r}_G$ is the virtual translation of the centre of mass: its variation will be zero under a rigid body virtual rotation about the centre of mass.

Letting $\delta\boldsymbol{\theta}_G$ be a rigid body virtual rotation of the system about its centre of mass, $\delta\boldsymbol{r} = \delta\boldsymbol{\theta}_{GA}\boldsymbol{r}$ and it is quite easy to show from (9) that the linear and angular momenta balance laws for the open system are respectively

$$\frac{d}{dt}\iiint_{R_o(t)} \rho\boldsymbol{u}\,dv = \boldsymbol{P} + \iint_{B_c(t)+B_o(t)} \bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\,ds + \iint_{B_o(t)} \rho\boldsymbol{u}(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds \tag{21}$$

and

$$\frac{d}{dt}\iiint_{R_o(t)} \rho\boldsymbol{r}_A\boldsymbol{u}\,dv = \iint_{B_c(t)+B_o(t)} \boldsymbol{r}_A\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\,ds + \iint_{B_o(t)} \rho(\boldsymbol{r}_A\boldsymbol{u})(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds\,. \tag{22}$$

These can be verified from the balance laws for the coincident material system on applying the general transport theorem (3). The last terms on the right of (21) and (22) represent momentum transport across the open surface $B_o(t)$.

7. The choice of control surface velocity

The control surface velocity chosen is either a matter of convenience or is governed by the nature of the problem under consideration. However, an intriguing possibility offers itself but one which, it would appear, is of limited use.

In the absence of any interaction of the system with its surroundings except over the open boundary $B_o(t)$, the energy equation (20) becomes

$$\frac{d}{dt}(K+E)_0 = \iint_{B_o(t)} \left[\boldsymbol{u}\cdot\bar{\boldsymbol{\sigma}} + \rho\left(\tfrac{1}{2}u^2+e\right)(\boldsymbol{V}-\boldsymbol{u})\right]\cdot\boldsymbol{n}\,ds \tag{23}$$

where the surface traction on the open surface may be non-conservative and depend upon displacement. Clearly, the total energy within the control volume will be conserved if the right side of (23) is zero; that is, if the open surface normal velocity is chosen as

$$\boldsymbol{V}\cdot\boldsymbol{n} = -\frac{\boldsymbol{u}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}}{\rho\left(\tfrac{1}{2}u^2+e\right)} + \boldsymbol{u}\cdot\boldsymbol{n}\,. \tag{24}$$

The corresponding Hamiltonian principle is now obtained from (9) by substitution as

$$\delta\int_{t_1}^{t_2}(K-E)_0\,dt + \int_{t_1}^{t_2} dt \iint_{B_o(t)} \left\{\delta\boldsymbol{r}\cdot\left[\bar{\boldsymbol{I}} - \frac{\boldsymbol{uu}}{\left(\tfrac{1}{2}u^2+e\right)}\right]\right\}\cdot\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{n}\,ds = 0 \tag{25}$$

where $\bar{\boldsymbol{I}}$ is the idemfactor.

Now, generally speaking, the Hamiltonian principle associated with conservative systems is an extremum principle and it might have been thought that the above pseudo-conservative system should behave in a similar fashion. Unhappily, this is not so and the reason can only be attributed to the fact that there is mass transport—we are not looking at the same particles at all times.

Having failed in this, is it at all possible by another choice of boundary velocity to make the Hamiltonian principle a stationary principle. The answer is in the affirmative—in some simple cases.

From (9) with no surface traction on the closed surface $B_c(t)$, a stationary principle would be obtained if $\delta \boldsymbol{r} \cdot [\bar{\sigma} + \rho \boldsymbol{u}(\boldsymbol{V} - \boldsymbol{u})] \cdot \boldsymbol{n}$ were zero over the open boundary $B_o(t)$. Excluding the trivial possibility of virtual displacements being zero everywhere on $B_o(t)$, this is only zero for one of the following conditions: that,

a) the dyadic $[\bar{\sigma} + \rho \boldsymbol{u}(\boldsymbol{V} - \boldsymbol{u})]$ is zero

b) the vector $[\bar{\sigma} + \rho \boldsymbol{u}(\boldsymbol{V} - \boldsymbol{u})] \cdot \boldsymbol{n}$ is zero

c) the vector $[\bar{\sigma} + \rho \boldsymbol{u}(\boldsymbol{V} - \boldsymbol{u})] \cdot \boldsymbol{n}$ is orthogonal to the virtual displacement $\delta \boldsymbol{r}$,

or,

d) the vector $\delta \boldsymbol{r} \cdot [\bar{\sigma} + \rho \boldsymbol{u}(\boldsymbol{V} - \boldsymbol{u})]$ is tangential to the open control surface.

The last two possibilities are thought to be impracticable since they either impose an additional unwelcome constraint on the virtual displacement field, or they imply the choice of a boundary and boundary velocity dependent on the virtual displacements. Accordingly, only the first two conditions are given consideration.

Notice that, without some conditions being imposed upon the stress and velocity fields $\bar{\sigma}$ and $\boldsymbol{u}$, the nine components of the dyadic cannot possibly be made zero by any choice of only the three components of the boundary velocity $\boldsymbol{V}$. A set of rectangular cartesian co-ordinates (1,2,3) is defined with the corresponding unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. The dyadic is identically zero provided only that each of the nine components $\sigma_{ij} + \rho u_i(V_j - u_j)$ is zero $(i, j = 1,2,3)$. The only possible solution for the open boundary velocity components satisfying this requirement can be shown to be

$$V_j = u_j - \frac{\sigma_{ij}}{\rho u_j} \tag{26}$$

with the necessary constraint on the stress and velocity fields that

$$\bar{\sigma}_A \boldsymbol{u} = 0 . \tag{27}$$

By similar reasoning, the vector $[\bar{\sigma} + \rho \boldsymbol{u}(\boldsymbol{V} - \boldsymbol{u})] \cdot \boldsymbol{n}$ is zero for the choice

$$\boldsymbol{V} \cdot \boldsymbol{n} = \boldsymbol{u} \cdot \boldsymbol{n} - \frac{\boldsymbol{e}_j \cdot \bar{\sigma} \cdot \boldsymbol{n}}{\rho u_j} . \tag{28}$$

but provided only that

$$\boldsymbol{n} \cdot \bar{\sigma}_A \boldsymbol{u} = 0 . \tag{29}$$

Consider the simple but commonly occurring case of a hydrostatic pressure p for which $\bar{\sigma} = -p\bar{I}$. Only condition (b) can be satisfied, although from (29) the velocity $\boldsymbol{u}$ is required to be normal to the control surface. Letting $\boldsymbol{u} = U\boldsymbol{n}$, it follows from (28) that the control surface normal velocity must be chosen as

$$\boldsymbol{V} \cdot \boldsymbol{n} = U + \frac{p}{\rho U} \tag{30}$$

to obtain the stationary form of the Hamiltonian principle

$$\delta \int_{t_1}^{t_2} (K - E)_0 \, dt = 0 . \tag{31}$$

The corresponding mass, momenta and energy balance equations read as

$$\frac{d}{dt}\iiint_{R_o(t)} \rho\, dv = \iint_{B_o(t)} \frac{p}{U}\, ds \tag{32}$$

$$\frac{d}{dt}\iiint_{R_o(t)} \rho \boldsymbol{u}\, dv = \boldsymbol{P} \tag{33}$$

$$\frac{d}{dt}\iiint_{R_o(t)} \rho \boldsymbol{r}_A \boldsymbol{u}\, dv = 0 \tag{34}$$

and

$$\frac{d}{dt}(K+E)_0 = -\iint_{B_o(t)} \frac{p}{U}(\tfrac{1}{2}U^2 - e)\, ds. \tag{35}$$

Having sacrificed energy conservation to obtain a stationary variational principle, it is indeed interesting to discover that we have gained instead the conservation of the linear and angular momenta of our system (provided the conservative body force $\boldsymbol{P}$ is zero). Inspection of equations (21) and (22) reveals that, in general, the momenta are conserved when either of the consitions (a) or (b) above is satisfied.

8. The rocket problem: An illustrative example

Consider the rectilinear motion of a rocket travelling in a vacuum. Its mass at any instant is m and the fuel burning rate β may be any prescribed function of time. The jet pressure p, density ρ and velocity $-v\boldsymbol{k}$ relative to the rocket casing are assumed uniform everywhere across the jet area A and are taken to be prescribed functions of time at any position along the jet; at the rocket nozzle their values are identified by the suffix "j". $\boldsymbol{k}$ is the unit vector in the direction of motion. For simplicity only (the variational statement (6) is valid for any degree of complexity), the centre of mass of the rocket is assumed to remain fixed in the rocket and the internal fuel flow effects are such as can be ignored. Gravitational effects are excluded for convenience.

The arrangement of the rocket and its control surfaces is shown below in Figure 2.

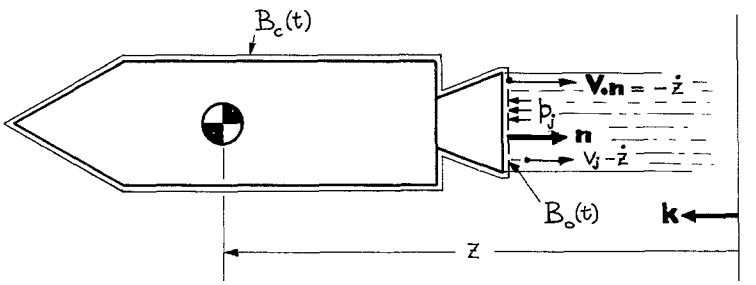

Figure 2.

The open and closed control surfaces $B_o(t)$ and $B_c(t)$ are attached to and move with the rocket. The rocket is subjected to a virtual displacement $\delta z\boldsymbol{k}$.

In the vacuum, there is no virtual work contribution from the closed surface but, at the open surface, the virtual work performed by the jet pressure is

$$\delta W_0 = \delta W = (pA)_j \delta z\,. \tag{36}$$

The jet particle velocity at the nozzle is $\boldsymbol{u}_j = (\dot{z} - v_j)\boldsymbol{k}$, the open surface velocity is $\boldsymbol{V} = \dot{z}\boldsymbol{k}$ and the normal $\boldsymbol{n} = -\boldsymbol{k}$. Thus, the appropriate statement of the extended Hamiltonian principle is

$$\delta \int_{t_1}^{t_2} \tfrac{1}{2} m\dot{z}^2\, dt + \int_{t_1}^{t_2} [(pA)_j - \beta(\dot{z} - v_j)]\delta z\, dt = 0 \tag{37}$$

where $\beta = (\rho v A)_j$ is the fuel burning rate. The position of the rocket is $z\boldsymbol{k}$ and its velocity is $\dot{z}\boldsymbol{k}$.

The system as defined moves as a rigid body with a velocity which is the velocity of the centre of mass and so we can write

$$\dot{z} = \frac{Dz}{Dt} \equiv \frac{dz}{dt} . \tag{38}$$

The virtual velocity is therefore

$$\delta\dot{z} = \frac{D\,\delta z}{Dt} \equiv \frac{d\,\delta z}{dt} . \tag{39}$$

In general however, $d(\ )/dt \neq D(\ )/Dt$.

Taking the indicated variation to the kinetic energy and employing the transport theorem,

$$\begin{aligned} \delta(\tfrac{1}{2}m\dot{z}^2) &= \frac{D}{Dt}(m\dot{z}\,\delta z) - \frac{D}{Dt}(m\dot{z}\,\delta z) \\ &= \frac{d}{dt}(m\dot{z}\,\delta z) + \beta\dot{z}\,\delta z - m\ddot{z}\,\delta z \end{aligned} \tag{40}$$

where $Dm/Dt = 0$ since the mass of the material system coinciding with the open system at time t must remain constant as the motion develops. The same result may be obtained as follows with the velocity $\dot{z}$ defined by the second form in (38),

$$\begin{aligned} \delta(\tfrac{1}{2}m\dot{z}^2) &= \frac{d}{dt}(m\dot{z}\,\delta z) - \frac{d}{dt}(m\dot{z})\delta z \\ &= \frac{d}{dt}(m\dot{z}\,\delta z) + \beta\dot{z}\,\delta z - m\ddot{z}\,\delta z \end{aligned}$$

since $dm/dt = -\beta$.

The virtual displacement δz is arbitrary, except at t_1 and t_2 when it is zero, and so it follows from (37) and (40) that

$$(pA)_j + \beta v_j = m\ddot{z} . \tag{41}$$

This is a well known form of the rocket equation [1] and could have been obtained directly from the momentum balance equation (21). The instantaneous mass is given by

$$m = M - \int_{t_1}^{t} \beta\, dt \tag{42}$$

where M is the initial mass at t_1.

9. Stationary principles for the rocket problem

Since the requisite conditions prevail—namely, a hydrostatic boundary pressure and a particle velocity normal to the open surface—it is possible to establish a stationary variational principle for this simple rocket problem by the choice of the jet control surface velocity suggested in (30) (there will however be an alternative choice available). A complication arises in that the Lagrangian must now include a contribution from a length of the ejected jet and we should expect the variational process to yield the equations of motion of the jet particles besides that for the rocket. The flow of the jet material in the vacuum is a difficult problem with the jet contained within a complicated shock formation and to obtain an expression for the internal energy of the Lagrangian would require knowledge of the relationship governing the changes in pressure, density and velocity. This lies outwith the scope of this simple illustrative example and to avoid the difficulty we assume that these are prescribed and known functions of time at all positions in the jet. It should not be thought however, that the variational principle is incapable of dealing with the detailed flow of the jet: provided all forms of energy of the jet were known, the equation of motion of the jet particles could be derived.

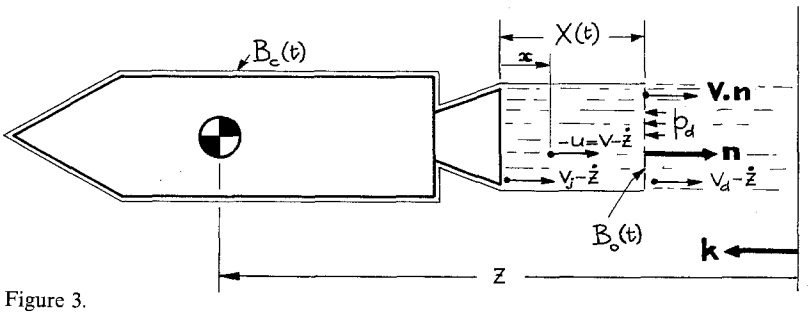

Figure 3.

The appropriate system is illustrated in Figure 3 above.
The stationary Hamiltonian principle is

$$\delta \int_{t_1}^{t_2} (L)_0 \, dt = 0 , \tag{43}$$

$$(L)_0 = \tfrac{1}{2} m\dot{z}^2 + \int_0^{X(t)} \tfrac{1}{2} \rho A u^2 \, dx \tag{44}$$

where all values are taken at time t. The length of the jet to be included in the open system is

$$X(t) = \int_{t_1}^{t} (\boldsymbol{V} \cdot \boldsymbol{n} + \dot{z}) dt \tag{45}$$

where the open surface normal velocity is obtained from (30), in terms of the absolute jet particle velocity $\boldsymbol{u} = (\dot{z} - v_d)\boldsymbol{k}$ at the open surface, as

$$\boldsymbol{V} \cdot \boldsymbol{n} = (v_d - \dot{z}) + \frac{p_d}{\rho_d (v_d - \dot{z})} . \tag{46}$$

The suffix "d" denotes values taken downstream at the jet open control surface.

Since the behaviour of the jet is prescribed, the only virtual displacement it can be subjected to is a rigid body translation in the line of motion and, moreover, if there is to be no discontinuity in the virtual displacement at the nozzle, the complete system—rocket and jet—must undergo the same rigid body virtual translation. Let is be $\delta z \boldsymbol{k}$.

Upon taking the indicated variations in (43) subject to the jet continuity conditions $D(\rho A)/Dt = 0$ and $\delta(\rho A) = 0$, remembering also that the portion of the open control volume defined by $0 \leqq x \leqq X(t)$ performs as a closed control volume to the material operators $D(\ )/Dt$ and $\delta(\ )$, we arrive at

$$\int_{t_1}^{t_2} \left\{ \frac{D}{Dt} \left[\delta z \left(m\dot{z} + \int_0^{X(t)} \rho A u \, dx \right) \right] - \delta z \frac{D}{Dt} \left[m\dot{z} + \int_0^{X(t)} \rho A u \, dx \right] \right\} dt = 0 . \tag{47}$$

Using the transport theorem to convert from material rates of change to rates of change appropriate to the open control volume on which the principle (43) was constructed, we get first

$$\int_{t_1}^{t_2} \left\{ \frac{d}{dt} \left[\delta z \left(m\dot{z} + \int_0^{X(t)} \rho A u \, dx \right) \right] - [\delta z \rho A u (\boldsymbol{V} - \boldsymbol{u}) \cdot \boldsymbol{n}]_d \right.$$
$$\left. - \delta z \frac{D}{Dt} \left[m\dot{z} + \int_0^{X(t)} \rho A u \, dx \right] \right\} dt = 0 \tag{48}$$

and then

$$\int_{t_1}^{t_1} \left\{ \frac{d}{dt} \left[\delta z \left(m\dot{z} + \int_0^{X(t)} \rho A u \, dx \right) \right] - \delta z \frac{d}{dt} \left[m\dot{z} + \int_0^{X(t)} \rho A u \, dx \right] \right\} dt = 0 . \tag{49}$$

From this, we obtain finally the equation of motion as

$$\frac{d}{dt}\left[m\dot{z} + \int_0^{X(t)} \rho A u\, dx\right] = 0\,, \quad u = \dot{z} - v \tag{50}$$

since δz is arbitrary in $t_1 < t < t_2$ and zero at t_1 and t_2. This states the conservation of linear momentum of the open system and agrees with (33) when body forces are excluded.

That (50) is equivalent to (41) can be demonstrated quite simply by obtaining the equation of momentum balance for the jet alone and substituting it in (47).

Earlier, it was mentioned that an alternative stationary variational statement could be obtained. This arises because the jet pressure distribution is prescribed and so under the virtual displacement δz, $p\delta z \equiv \delta(pz)$ enabling us to write the extended principle (9) as

$$\delta \int_{t_1}^{t_2} [(L)_0 + (Ap)_d z]\, dt = 0 \tag{51}$$

provided a closed control volume is employed with the control surface velocity that of the local jet particles. Suffix "d" refers to the downstream closed control surface. The Lagrangian has the same form as (44) but the length of the jet is now

$$X(t) = \int_{t_1}^{t} (V \cdot \boldsymbol{n} + \dot{z})\, dt\,, \quad V \cdot \boldsymbol{n} = v_d - \dot{z}\,. \tag{52}$$

The linear momentum of this material system will not be constant but will increase at the rate $(Ap)_d$ which is the nett force on the jet closed control surface. The equation of motion derived from (51) is

$$\frac{D}{Dt}\left(m\dot{z} + \int_0^{X(t)} \rho A u\, dt\right) = (Ap)_d\,. \tag{53}$$

Notice that the identical result could have been obtained from (46) and (48) if the conditions of δz being zero at t_1 and t_2 and arbitrary at all other times had been imposed at that stage.

10. Some comments on the application to the dynamics of flexible pipes conveying fluid

Housner [5] tacitly assumed the applicability of Hamilton's principle to the flow of liquid through flexible pipes having prescribed end conditions. Although it transpires that he assumed correctly and proceeded to obtain the correct partial differential equation of motion, Benjamin [6] was of the opinion that the derivation was erroneous.

The latter author investigated the dynamics of articulated pipes conveying fluid and having one end free—a much more difficult problem. He obtained the equations of motion by a partly intuitive process and later deduced the form of the Hamiltonian principle for the limiting case of a continuous, flexible, cantilevered tube. This is:

$$\delta \int_{t_1}^{t_2} (K - E)_0\, dt - \int_{t_1}^{t_2} MU(\dot{\boldsymbol{R}} + U\boldsymbol{\tau}) \cdot \delta \boldsymbol{R}\, dt = 0 \tag{54}$$

where $(K)_0$ is the kinetic energy of the pipe and fluid, $(E)_0$ is the strain energy of the pipe, M is the mass of fluid contained per unit length, U its speed, $\boldsymbol{R}$ is the position vector of the exit to the pipe (assumed free) and $\boldsymbol{\tau}$ is the outward normal to the pipe exit. Although the system is open, its mass remains constant. The liquid is incompressible and of density ρ.

This principle can be verified independently using the results of the present work.

First of all, the system is open with the control volume surface coinciding with the exterior surface of the pipe and the pipe inlet and exit. The arrangement is shown in Figure 4. The cross-sectional area of the pipe is a.

To reproduce (54) from the general principle (9), we need concentrate attention only on the verification of the second integral of (54).

Conditions are assumed uniform over the cross-section of the pipe. S_i and $S_e(t)$ are the open control surfaces at the pipe inlet and exit respectively. The pipe root is fixed and so there can be

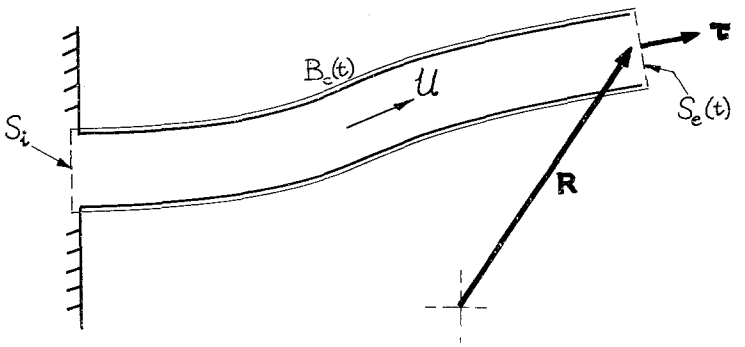

Figure 4.

no virtual work contribution from the forces and moments of reaction. Consequently, in the extended principle (6),

$$\delta H = -\iint_{B_c(t)+S_i+S_e(t)} p(\delta \boldsymbol{r}\cdot\boldsymbol{n})ds + \iint_{S_i+S_e(t)} \rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds \tag{55}$$

The system is comprised of the pipe and the fluid contained within the control volume and although there is no virtual displacement of the pipe root, nothing has been said yet to suggest that the fluid particles cannot suffer a virtual displacement there along the axis of the pipe—hence the inclusion of the surface S_i in the virtual momentum integral in (54) above.

The pipe is assumed to behave as a simple beam in bending with its cross-sectional area, and hence volume, remaining unchanged by the virtual displacement. In addition, the generalized co-ordinates of the pipe characterize the system and a generalized virtual displacement does not (in this case) induce a virtual displacement of the fluid relative to the pipe. Thus, any virtual displacement of the fluid relative to the pipe will be arbitrary and independent of the pipe's virtual displacement. It follows therefore that there can be no virtual change to the volume of the system since the fluid is incompressible and we can write

$$\begin{aligned} -\iint_{B_c(t)+S_i+S_e(t)} p(\delta\boldsymbol{r}\cdot\boldsymbol{n})ds + \iint_{S_i} \rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds = \\ -\iint_{S_i}(p-p_\infty)(\delta\boldsymbol{r}\cdot\boldsymbol{n})ds + \iint_{S_i}\rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds\,. \end{aligned} \tag{56}$$

This will be zero if the fluid behaviour at the inlet S_i is prescribed. We follow Benjamin in doing this since to do otherwise would complicate unnecessarily the variation to the Lagrangian.

The fluid velocity at the pipe exit is $\boldsymbol{u}=\dot{\boldsymbol{R}}+U\boldsymbol{\tau}$ and the normal velocity relative to the control surface $S_e(t)$ is $(\boldsymbol{u}-\boldsymbol{V})\cdot\boldsymbol{n}=U$. Thus, (55) finally becomes

$$\begin{aligned} \delta H &= \iint_{S_e(t)} \rho(\boldsymbol{u}\cdot\delta\boldsymbol{r})(\boldsymbol{V}-\boldsymbol{u})\cdot\boldsymbol{n}\,ds \\ &= -MU(\dot{\boldsymbol{R}}+U\boldsymbol{\tau})\cdot\delta\boldsymbol{R}\,, \quad M=\rho a \end{aligned} \tag{57}$$

and proves Benjamin's statement of the Hamiltonian principle. On the other hand, this is zero for a supported end because $\delta\boldsymbol{R}=0$ and so lends some support for Housner's assumption of the applicability of Hamilton's principle to the supported pipe. There is a difficulty yet, however.

The crux of Benjamin's disagreement with Housner's analysis lies in a fundamental difference between the cantilevered pipe and a pipe with simply-supported or fixed ends. Within the limitations of small transverse displacements, the cantilevered pipe is assumed infinitely stiff to longitudinal extension and maintains a constant length as it bends, whereas the supported or fixed pipe must necessarily increase in length on bending and is taken to be infinitely flexible in the longitudinal direction. In neither case is there a strain energy contribution from longitudinal

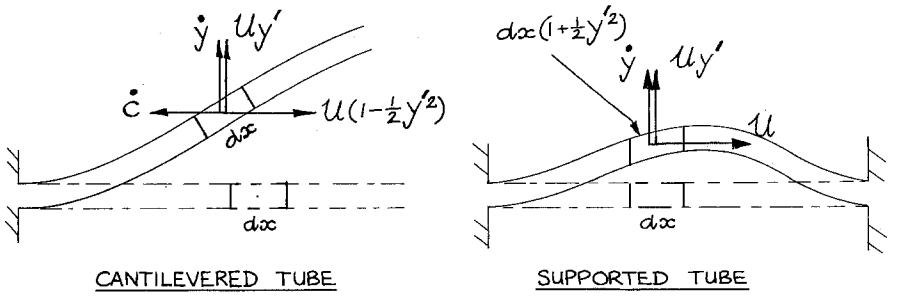

Figure 5.

effects. Now, Housner's Lagrangian is perfectly correct provided the local cross-sectional area of the supported pipe diminishes on bending to permit satisfaction of the requirements of incompressibility of the system and of a constant rate of mass flow through the pipe. Unfortunately, these points were not established by Housner. They are considered below and his Lagrangian demonstrated to be correct.

Elemental lengths dx of a cantilevered and a supported pipe are compared in the deformed conditions in Figure 5 above.

With the latter, the cross-sectional area reduced from a to $a\,[1-\frac{1}{2}(y')^2]$ to maintain the fluid volume and the fluid velocity relative to the pipe increases to $U[1+\frac{1}{2}(y')^2]$ to maintain locally the rate of mass flow. The respective horizontal and vertical fluid velocity components are shown correct to second order of small quantities. The lateral displacement is y with a gradient $y' = \partial y/\partial x$ and the longitudinal displacement of a point on the cantilevered pipe (assumed to be zero on the supported pipe) is

$$c(x) = \int_0^x \tfrac{1}{2}(y')^2\,dx\,. \tag{58}$$

As a consequence, the fluid kinetic energy is

$$K_{\text{fluid}} = \int_0^l \tfrac{1}{2}M[(\dot{y}+Uy')^2+U^2]\,dx\,, \quad \dot{y} = \frac{\partial y}{dt} \tag{59}$$

for the supported tube and is in agreement with Housner's form; whereas, for the cantilevered tube,

$$K_{\text{fluid}} = \int_0^l \tfrac{1}{2}M[(\dot{y}+Uy')^2+U^2-U^2(y')^2-2\dot{c}U]\,dx \tag{60}$$

as obtained by Benjamin. These are correct to second order of small quantities. l denotes the equilibrium length of the pipes.

Benjamin was quite correct to be wary of attempting to analyse the system in terms of its overall energy. For, without this extended form of the Hamiltonian principle, the system control volume had to be closed (or, as only now demonstrated in this paper, where the virtual displacements on the open control surface are zero). To ensure this would require the inclusion within the control volume of the accumulated discharge from the free end and the infinite reservoir of fluid required to maintain a constant mass flow for an indefinitely long time. As a result, the total energy of the system would be infinite and the Hamiltonian principle for the material system largely meaningless.

The introduction of the open control volume has thus freed our choice of the system boundaries—it is now possible to ensure that the total energy is bounded and to include within the system only those portions making a direct contribution to the problem in hand.

Herrmann and Nemat-Nasser [7] extend Benjamin's analysis to the investigation of the instability modes of cantilevered bars induced by fluid flow through attached pipes. The bars have freedom in flexure and torsion.

11. Discussion

The extended Hamiltonian principle developed herein, is applicable to a wide range of problems subject only to the definition of suitable expressions for the Lagrangian and virtual work contributions. Amongst these are the more sophisticated dynamics of flexible rockets, the extrusion process, the translating catenary and the motion of ablating vehicles—to mention but a few. The principle can be employed to obtain the differential equations of motion of the system or as the basis for their approximate solution.

The ability to choose the motion of the system control surface boundary to suit the problem in hand, rather than have it prescribed by the necessity of its being a material surface (or a surface on which the virtual displacements are zero), is an important advance. This is so particularly with fluid fields where attention is fixed preferably on conditions at points in space (Eulerian description) rather than on specific particles (Lagrangian description). The resulting loss of particle identity means that the position of any material surface as it deforms is difficult to obtain. It is for this reason that fluid systems are frequently analysed employing fixed (open) control surfaces: usually it is simpler to determine the velocity at the fixed surface, and so the nett outflow, than to calculate the particle paths.

This paper establishes the variational counterpart of the Newtonian treatment of variable mass systems.

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